SM358

Frequently Asked Questions

Can I take an operator outside a Dirac bracket?

No, definitely not! Think what $\langle f|\widehat{\mathrm{B}}|g\rangle$ means as an integral.

In one dimension, the operator $\widehat{\mathrm{B}}$ acts on the function $g(x)$, and the result is multiplied by $f^*(x)$ to give an integrand which is then integrated over all x:

$$\langle f|\widehat{\mathrm{B}}|g\rangle = \int_{-\infty}^{\infty} f^*(x)\,\widehat{\mathrm{B}}g(x)\,\mathrm{d}x.$$

The result is a number (in general complex), the value of the integral.

Now, if you take the operator outside $\langle f|\widehat{\mathrm{B}}|g\rangle$ you get $\widehat{\mathrm{B}}\langle f|g\rangle$. This step not justified but let us see what $\widehat{\mathrm{B}}\langle f|g\rangle$ would mean.

Note that $\langle f|g\rangle$ is a shorthand for the definite integral $\int_{-\infty}^{\infty} f^*(x)g(x)\,dx$ and is therefore a number. So $\widehat{\mathrm{B}}\langle f|g\rangle$ is the result of acting with the operator $\widehat{\mathrm{B}}$ on the constant function $\langle f|g\rangle$. This is not at all related to the original $\langle f|\widehat{\mathrm{B}}|g\rangle$, where $\widehat{\mathrm{B}}$ acted on $g(x)$ before the integration.

For example, $\widehat{\mathrm{p}}_x\langle f|g\rangle$ is always equal to 0 because the derivative of any constant is zero. The same cannot be said of $\langle f|\widehat{\mathrm{p}}_x|g\rangle$.

To take an operator outside a Dirac bracket is as bad as simplifying $\int_0^{\pi/3} x^2\sin(x)\,dx$ by pretending that this is equal to $x^2 \times \int_0^{\pi/3}\sin(x)\,dx$. Its just something that should not be done.

Of course, if $g(x)$ is an eigenvector of $\widehat{\mathrm{B}}$, so that $\widehat{\mathrm{B}}|g\rangle = \lambda|g\rangle$, where the eigenvalue λ is a number, then we can say

$$\langle f|\widehat{\mathrm{B}}|g\rangle = \langle f|\lambda|g\rangle = \lambda\langle f|g\rangle,$$

but this is the ONLY case in which this can be done. It works because constants that are initially inside an integral can be moved outside the integral.

It may help to remember that operators act on functions to create other functions – or on kets to produce other kets. They are conceptually different to functions or constants.

It may also help to remember that any completed Dirac bracket such as $\langle f|g\rangle$ or $\langle f|\widehat{\mathrm{B}}|g\rangle \equiv \langle f|\widehat{\mathrm{B}}g\rangle$ is a number (in general, complex).

Copyright © 2011 The Open University

We sometimes see mistakes made in exam questions. For example, the Hermitian property for an operator $\widehat{\mathrm{B}}$ states that

$$\langle f|\widehat{\mathrm{B}}g\rangle = \langle\widehat{\mathrm{B}}f|g\rangle$$

for any normalizable functions f and g. The requirement that the functions f and g be normalizable is a technicality, needed for the definition to work under all the circumstances we need (e.g. needed to establish that the momentum operator is Hermitian.) Unfortunately, we have seen many answers where the idea of normalization has been seized on and made to work in shady ways. For example, if ψ is normalized, we sometimes see the following line of reasoning:

$$\langle\psi|\widehat{\mathrm{B}}|\psi\rangle = \widehat{\mathrm{B}}\langle\psi|\psi\rangle = \widehat{\mathrm{B}}\,1 = \widehat{\mathrm{B}}. \qquad \text{(WRONG!)}$$

This is nonsense because it says that the *number* $\langle\psi|\widehat{\mathrm{B}}|\psi\rangle$ is equal to the *operator* $\widehat{\mathrm{B}}$, which is like equating chalk and cheese. Two things have gone wrong here: the operator cannot be lifted outside the Dirac bracket, and $\widehat{\mathrm{B}}$ acting on the number 1 does not necessarily give $\widehat{\mathrm{B}}$. (For example, if $\widehat{\mathrm{B}}$ is a multiple of the differentiation operator, we have $\widehat{\mathrm{B}}\,1 = 0$, though this does not rescue the above conclusion because the first error still stands.)

The fact that you cannot take operators outside integrals is a strong restriction, and limits and you can do with them. This is good news because it limits the manipulations you can try when attempting to prove something. The most important properties of operators in QM are linearity and the fact that the operators of observable quantities are Hermitian. Naturally, the Hermitian property involves shuffling operators around inside integrals, and not taking then outside.